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ON INVARIANT ADDITIVE SUBGROUPS

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ABSTRACT

Suppose that R is a prime ring with the center Z and the extended centroid C. An additive subgroup A of R is said to be invariant under special automorphisms if $(1+t)A(1+t)^{-1} \subseteq A$ for all $t \in R$ such that $t^2 = 0$. Assume that R possesses nontrivial idempotents. We prove: (1) If ch $R \neq 2$ or if $RC \neq C_2$, then any noncentral additive subgroup of R invariant under special automorphisms contains a noncentral Lie ideal. (2) If ch $R = 2$, $RC = C_2$ and $C \neq \{0, 1\}$, then the following two conditions are equivalent: (i) any noncentral additive subgroup invariant under special automorphisms contains a noncentral Lie ideal; (ii) there is $\alpha \in Z \setminus \{0\}$ such that $\alpha^2 Z \subseteq \{\beta^2 : \beta \in Z\}$.

The aim of this paper is to prove a conjecture raised in Herstein [4]. In [4], both the main theorem and the conjecture are stated under the assumption that R possesses a nontrivial idempotent. But it seems more useful to formulate them under the weaker assumption that Q, the two sided Martindale quotient ring as defined on p. 156 of [5], possesses a nontrivial idempotent. The main theorem of [4] and all its lemmas, together with their proofs, remain valid word for word under this context. So we will assume all lemmas of [4] in this strength.

We recall some definitions and meanwhile fix our notation. In what follows, R will be a prime ring and Z its center. The *left* Martindale quotient ring of R is a ring S satisfying the following axioms (p. 156 [5] and also p. 20 [3]):

(1) $R \subseteq S$.

(2) For each $a \in S$, there is a nonzero ideal I of R such that $Ia \subseteq R$.

(3) If $a \in S$ and $Ia = 0$ for some nonzero ideal I of R, then $a = 0$.

(4) Let I be a nonzero ideal of R. If $\phi : {}_RI \to {}_RR$ is a left R-module homomorphism then there exists $a \in S$ such that $i\phi = ia$ for all $i \in I$.

The *two-sided* Martindale quotient ring of R is the subring Q of S consisting

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of those elements $a \in S$ for which there exists a nonzero ideal I of R such that $aI \subseteq R$. The center of Q, denoted by C, is called the extended centroid of R. Note that the centers of Q and S coincide.

By a Lie ideal of R, we mean an additive subgroup U of R such that $U \supseteq [U, R]$. For convenience, we call a Lie ideal U *proper* if there is a nonzero ideal I of R such that $U \supseteq [I, R]$. A Lie ideal U is proper if and only if $[U, U] \neq 0$. If R is not commutative, then for any *proper* Lie ideal U, $\bar{U}$, the subring generated by U, always contains a nonzero ideal of R. Also, if $aUb = 0$, then $a = 0$ or $b = 0$ (see [2]). Since Q will be assumed to possess a nontrivial idempotent throughout, R cannot be commutative and the above two properties will hold always.

An additive subgroup A of R is said to be invariant under all special automorphisms in R if $(1+t)A(1+t)^{-1} \subseteq A$ for all $t \in R$ such that $t^2 = 0$. Our main theorem is the following:

THEOREM 1. *Assume that Q possesses a nontrivial idempotent $e = e^2 \neq 0, 1$ and that A is an additive subgroup of R invariant under all special automorphisms of R. Then either $A \subseteq Z$ or A contains a proper Lie ideal of R, unless* $\operatorname{ch} R = 2$ *and* $\dim_C RC = 4$.

From now on, we assume that Q possesses nontrivial idempotents. We begin our proof with a reformulation of lemma 6 [4].

LEMMA 1. *Assume that A is an additive subgroup of R invariant under all special automorphisms of R. If there exist a nontrivial idempotent e and a nonzero ideal I of R such that $[e, I] \subseteq A$, then A contains a proper Lie ideal of R.*

PROOF. Let $W = \{r \in R : [r, I] \subseteq A\}$. Then W is a subgring of R. Let J be a nonzero ideal of R such that $eJ \subseteq R$, $Je \subseteq R$ and $eJe \subseteq R$. Let $t \in eJ(1-e) \subseteq R$. Then $et = t$ and $te = t^2 = 0$. Since $[e, I] \subseteq A$ and A is invariant under special automorphisms of R, we have $[(1-t)e(1+t), I] \subseteq A$. Hence $t = et - te - tet \in W$. So $eJ(1-e) \subseteq W$. Similarly $(1-e)Je \subseteq W$. Since W is a subring, $(1-e)JeJ(1-e) \subseteq W$. Set $V = JeJ$. Then $(1-e)V(1-e) \subseteq W$. Since $V \subseteq J$, we also have $eV(1-e) \subseteq eJ(1-e) \subseteq W$. Hence $V(1-e) \subseteq (1-e)V(1-e) + eV(1-e) \subseteq W$. Similarly, $(1-e)V \subseteq W$. So W contains the nonzero ideal $K = V(1-e)V$. Thus $A \supseteq [K, I]$. This implies immediately that A contains a proper Lie ideal of R.

Let E be the additive subgroup generated by nontrivial idempotents of Q, and T, the additive subgroup generated by those $t \in R$ such that $t^2 = 0$. The following lemma collects two well-known facts about E and T.

LEMMA 2. (1) *E is a proper Lie ideal of Q and T contains a proper Lie ideal of R.*

(2) *For $a \in Q$, if $[a, E] = 0$ or $[a, T] = 0$, then $a \in C$.*

PROOF. (1) It is well-known that E is a Lie ideal of Q. Now we show that E is proper. Suppose otherwise. Then $[E, E] = 0$. Let $e = e^2 \neq 0, 1$. For $x \in Q$, $e + ex(1-e)$ is also a nontrivial idempotent. Hence $e + ex(1-e) = e(e + ex(1-e)) = (e + ex(1-e))e = e$. So $ex(1-e) = 0$. Since Q is also prime, this implies $e = 0$ or $1 - e = 0$. This is absurd!

Now we show that T contains a proper Lie ideal of R. Let e be a nontrivial idempotent of Q. Pick a nonzero ideal I of R such that $eI \subseteq R$, $Ie \subseteq R$ and $eIe \subseteq R$. Then for $x \in I$, we have $ex(1-e)$, $(1-e)xe \in T$. Hence $[e, x] = ex(1-e) - (1-e)xe \in T$. That is $[e, I] \subseteq T$. T is obviously invariant under all special automorphisms of R. So Lemma 1 gives the desired result.

(2) If $[a, E] = 0$, then $[a, \bar{E}] = 0$. Since $\bar{E}$ contains a nonzero ideal of Q, $a \in C$. Similarly for $[a, T] = 0$.

We assume from now on that $\operatorname{ch} R \neq 2$ *or* $\dim_C RC > 4$ and that A is a *noncentral* additive subgroup of R invariant under all special automorphisms of R.

LEMMA 3. *For $x, y \in Q$, if $xAy = 0$, then $x = 0$ or $y = 0$.*

PROOF. Without loss of generality, we may assume that $x = y \in R$: For given $x, y \in Q$, we can find a nonzero ideal I of R such that $yIx \subseteq R$. For $c \in yIx$, we have $cAc = 0$. Suppose that we could show $c = 0$. Then we have $yIx = 0$ and this implies $y = 0$ or $x = 0$.

So now we assume that $c \in R$ is such that $cAc = 0$. Let $\bar{A}$ be the subring generated by A. We *claim* that $c\bar{A}c = 0$.

First assume that $c^2 \neq 0$. Let $b \in A$ and $r \in R$. Set $t = bcrc$. Then $ct = 0 = t^2$. For $a \in A$, $at - ta + tat \in A$ and hence

$$0 = c(at - ta + tat)c = catc = cabcrc^2.$$

Since $c^2 \neq 0$, we have $cabc = 0$. That is $cA^2c = 0$. Set $B = A + A^2$. Then B is also an additive subgroup of R invariant under special automorphisms of R and also $cBc = 0$. Repeating the above argument for B in place of A, we have $c(B + B^2)c = 0$. That is $c(A + A^2 + A^3 + A^4)c = 0$. Continuing in this manner, we have $c\bar{A}c = 0$.

Now assume that $c^2 = 0$. Let $a \in A$, $r \in R$. Set $t = crca$. Then $t^2 = 0$ and $ct = 0 = tc$. Let $b \in A$. Then $m = tb - bt + tbt \in A$. Note that $cm = 0$ and

$mc = tbc = crcabc$. Hence

$$crcabc = mc = mc - cm + cmc \in A.$$

Set $l = crcabc$. Note that $lc = cl = 0$. For $s \in R$ such that $s^2 = 0$, we have

$$0 = c(ls - sl + sls)c = cslsc = cscrcabcsc.$$

So $cabcsc = 0$ always. That is $cabcTc = 0$. By Lemma 9 (p. 120 [6]), this implies $cabc = 0$. That is $cA^2c = 0$. Arguing as in the previous case, we have $c\bar{A}c = 0$ again.

Now using the claim and applying lemma 5 [4] to $\bar{A}$, we have $c = 0$ as desired.

The following is crucial.

LEMMA 4. *Suppose that $e \in Q$ is a nontrivial idempotent such that $\dim_C(1-e)ACe > 1$. Then there is a nonzero ideal I of R such that $eI(1-e) \subseteq A$.*

PROOF. Let V be a nonzero ideal of R such that $eV \subseteq R$, $Ve \subseteq R$ and $eVe \subseteq R$. For $s, t \in eV(1-e)$ and $a \in A$, we have

$$sa - as + sas \in A, \quad ta - at + tat \in A$$

and

$$(s+t)a - a(s+t) + (s+t)a(s+t) \in A.$$

Subtracting the first two formulae from the last one, we have $sat + tas \in A$. Writing $s = eu(1-e)$ and $t = ev(1-e)$ for $u, v \in V$, we have

$$eu(1-e)aev(1-e) + ev(1-e)aeu(1-e) \in A.$$

Let us define $B = \{x \in R: ex(1-e) \in A\}$. From the above, we have $ubv + vbu \in B$ for any $b \in (1-e)Ae$ and for any $u, v \in V$.

Now by our assumption that $\dim_C(1-e)ACe > 1$, we can choose $b, c \in (1-e)Ae$ such that b, c are C-independent. We *claim* that there exist $u_1, v_1, \dots, u_n, v_n \in V$ such that

$$u_1bv_1 + \cdots + u_nbv_n = 0$$

but

$$u_1cv_1 + \cdots + u_ncv_n \neq 0.$$

Suppose otherwise. Then for any $u_1, v_1, \dots, u_n, v_n \in V$,

$$(*) \qquad u_1bv_1 + \cdots + u_nbv_n = 0 \quad \text{implies} \quad u_1cv_1 + \cdots + u_ncv_n = 0.$$

We define a map α: $VbV \to VcV$ by

$$\alpha(u_1bv_1 + \cdots + u_nbv_n) = u_1cv_1 + \cdots + u_ncv_n$$

for $u_1, v_1, \ldots, u_n, v_n \in V$. By (*), α is well-defined. It is obvious that α is a R-bi-homomorphism. So $\alpha \in C$ and $\alpha b = c$. This contradicts the C-independence of b, c.

By the claim above, let us fix $u_1, v_1, \ldots, u_n, v_n \in V$ such that

$$u_1bv_1 + \cdots + u_nbv_n = 0 \quad \text{but} \quad u_1cv_1 + \cdots + u_ncv_n = p \neq 0.$$

For $x, y \in R$,

$$v_1ybxu_1 + \cdots + v_nybxu_n = v_1ybxu_1 + \cdots + v_nybxu_n + (xu_1bv_1y + \cdots + xu_nbv_ny) \in B.$$

(The equality above follows from the fact $xu_1bv_1y + \cdots + xu_nbv_ny = 0$.) Set $J = RbR$. The above says for $r \in J$,

$$v_1ru_1 + \cdots + v_nru_n \in B.$$

For $x, y \in V^2JV^2 \subseteq R$, we also have

$$xu_1cv_1y + \cdots + xu_ncv_ny + v_1ycxu_1 + \cdots + v_nycxu_n \in B.$$

Since $ycx \in J$, $v_1ycxu_1 + \cdots + v_nycxu_n \in B$ by the paragraph above. We have

$$xu_1cv_1y + \cdots + xu_ncv_ny = xpy \in B.$$

Thus $(V^2JV^2)p(V^2JV^2) \subseteq B$ and hence $eI(1-e) \subseteq A$ where I is the nonzero ideal $(V^2JV^2)p(V^2JV^2)$.

LEMMA 5. *Let e be a nontrivial idempotent of Q. If $\dim_C(1-e)ACe = 1$, then Q satisfies a G.P.I. and hence has a minimum idempotent.*

PROOF. By Lemma 3, we can pick $a \in A$ such that $ea(1-e) \neq 0$. Let

$$b(u, v) = (1-e)uea(1-e)ve + (1-e)vea(1-e)ue.$$

Choose a nonzero ideal V of R such that $eV \subseteq R$, $Ve \subseteq R$ and $eVe \subseteq R$. For $u, v \in V$, by the argument in Lemma 4, $b(u, v) \in A$ and hence $b(u, v) = (1-e)b(u, v)e \in (1-e)Ae$. Since $\dim_C(1-e)ACe = 1$,

$$b(u_1, v_1)xb(u_2, v_2) = b(u_2, v_2)xb(u_1, v_1)$$

holds for $u_1, v_1, u_2, v_2, x \in V$. This is a nontrivial G.P.I. for V. So V satisfies a G.P.I. and hence so does Q. By [7], Q has a minimum idempotent.

LEMMA 6. *If e is a minimum idempotent of Q, then* $\dim_C eA(1-e)>1$ *except when* $RC = C_2$.

PROOF. Set $M = eQ$ and $D = eQe$. Q, via right multiplication, can be regarded as a dense subring of $\mathrm{Hom}({}_DM, {}_DM)$. Choose a basis $\{v_1, v_2, \ldots\}$ for M such that $v_1e = v_1$ but $v_2e = v_3e = \cdots = 0$. Pick $a \in A$ such that $ea(1-e)\neq 0$ and let $u = v_1ea(1-e)$. Observe that $u\neq 0$, for otherwise we would have $ea(1-e)=0$ by the faithfulness of such representation. Let $b \in Q$ be such that $ub = 0$. Then $v_iea(1-e)b = 0$ for all i. Hence $ea(1-e)b = 0$. Assume that $\dim_C eAC(1-e) = 1$. Then we have $eA(1-e)b = 0$. By Lemma 3, $(1-e)b = 0$. Thus $v_2b = v_2eb = 0$, $v_3b = v_3eb = 0, \ldots$. Since this holds for any $b \in Q$ such that $ub = 0$, Jacobson's density theorem says that $v_2, v_3, \ldots$ are all D-dependent on u. Hence $\dim_D M = 2$ and $Q = D_2$. We must show $D = C$.

Write

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Fix a nonzero element $a = \left(\begin{smallmatrix} 0 & 0 \\ x & 0 \end{smallmatrix}\right)$ of $(1-e)Ae$. Let

$$s = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, \qquad t = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$$

be two elements of $eQ(1-e)\cap R$. We compute

$$sat + tas = \begin{pmatrix} 0 & uxv + vxu \\ 0 & 0 \end{pmatrix} \in eA(1-e).$$

Hence the set

$$\left\{ uxv + vxu \colon \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in eQ(1-e)\cap R \right\}$$

is one dimensional over C. By Lemma 5, Q satisfies a nontrivial G.P.I. By the main result of Martindale [7], D is finite dimensional over C. Since $Q = D_2$, R is a P.I. ring and hence $Q = RC$ is simply the localization of R at its center Z. Given $u, v \in D$, we can find $\alpha, \beta \in Z\setminus\{0\}$ such that

$$\begin{pmatrix} 0 & \alpha u \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \beta v \\ 0 & 0 \end{pmatrix}$$

are elements of $eQ(1-e)\cap R$. Hence the set $S = \{uxv + vxu \colon u, v \in D\}$ is also of dimension 1 over C.

If $\mathrm{ch}\, R\neq 2$, let $v = \frac{1}{2}x^{-1}$. Then $uxv + vxu = u$. So D is one dimensional over C and $D = C$. So we assume $\mathrm{ch}\, R = 2$. For $u \in D$, note that $[u, x] = ux1 + 1xu \in S$.

If $x \notin C$, choose $u \in D$ such that $0 \neq [u,x]$. Then $[ux,x] = \lambda[u,x]$ for some $\lambda \in C$. But this implies $x = \lambda \in C$, absurd! So $x \in C$. Replacing v by $x^{-1}v$ in $uxv + vxu$, we obtain that $\{uv + vu : u, v \in D\}$ is one dimensional over C. Suppose that $[u,v] \neq 0$. Then $[u, vu] = \lambda[u,v]$ for some $\lambda \in C$. But this implies $u = \lambda \in C$, a contradiction again. So D is commutative and hence $D = C$.

LEMMA 7. *If $RC \neq C_2$, then A contains a proper Lie ideal.*

PROOF. By Lemma 1, it suffices to show that there exist a nontrivial idempotent e and a nonzero ideal I of R such that $[e, I] \subseteq A$.

Suppose that Q has a minimum idempotent e. Since we assume that $RC \neq C_2$, by Lemma 6, both $eAC(1-e)$ and $(1-e)ACe$ have C-dimension >1. By Lemma 4, there exists a nonzero ideal I of R such that $eI(1-e) \subseteq A$ and $(1-e)Ie \subseteq A$. Hence $[e,I] = eI(1-e) - (1-e)Ie \subseteq A$.

Now suppose that Q has no minimum idempotents. By Lemma 5, $\dim_C(1-e)ACe > 1$ for any nontrivial idempotent e. Arguing as in the previous paragraph, we have $[e,I] \subseteq A$ for some nonzero ideal I of R.

PROOF OF THEOREM 1. Now we are left with the case $RC = C_2$. By the assumption, $\operatorname{ch} R \neq 2$. As remarked in [4], the argument below, essentially the one given by Amitsur [1], actually gives a proof of Theorem 1 when $\operatorname{ch} R \neq 2$.

Suppose $\operatorname{ch} R \neq 2$. Let $a \in A$ and $t \in T$. Then

$$(1-t)a(1+t) - (1+t)a(1-t) = 2(at - ta) = 2[a,t] \in A.$$

So we have $[2T, A] \subseteq A$. By Lemma 2, T contains a proper Lie ideal of R and hence so does $2T$. By theorem 13 (p. 123, [6]), A also contains a proper Lie ideal of R. This completes the proof of Theorem 1.

Surprisingly, unlike the main theorem in [4], Theorem 1 is actually false in general when $\operatorname{ch} R = 2$, $RC = C_2$ and $C \neq \{0,1\}$. Instead of constructing a single counterexample, we would like to give a thorough analysis of this case.

Throughout the rest of the paper, we assume that $\operatorname{ch} R = 2$, $RC = C_2$ and $C \neq \{0,1\}$. Define $Z^0 = \{\alpha^2 : \alpha \in Z\}$, where Z is the center of R as before. Following (p. 271 [9]), Z is said to be a fractionary ideal of Z^0 if and only if there is $0 \neq \alpha \in Z^0$ such that $\alpha Z \subseteq Z^0$. Our next objective is to prove

THEOREM 2. *Suppose that $RC = C_2$, $\operatorname{ch} R = 2$ and $C \neq \{0,1\}$. Then the following two conditions are equivalent:*

(1) *Every noncentral additive subgroup of R invariant under all special automorphisms contains a proper Lie ideal of R.*

(2) *Z is a fractionary ideal of Z^0.*

PROOF OF (1) $\Rightarrow$ (2). Let a_{ij} $(i, j = 1, 2)$ be the matrix units of C_2 such that $e_{ij}e_{kl} = \delta_j^k e_{il}$ and $e_{11} + e_{22} = 1$. Since RC is simply the localization of R at its center Z, there is $\beta \in Z$ such that $\beta e_{ij} \in R$ for all $1 \leqq i,\ j \leqq 2$. Suppose that

$$r = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \sum_{i,j} \alpha_{ij} e_{ij} \in R, \qquad \text{where } \alpha_{ij} \in C.$$

Then $\alpha_{ij} = \alpha_{ij} 1 = e_{1i} r e_{j1} + e_{2i} r e_{j2}$. Hence

$$\beta^2 \alpha_{ij} = (\beta e_{1i}) r (\beta e_{j1}) + (\beta e_{2i}) r (\beta e_{j2}) \in R \cap C = Z.$$

Let Z_1 be the additive subgroup of C generated by those α_{ij} $(i, j = 1, 2)$ such that $\sum_{i,j=1}^2 \alpha_{ij} e_{ij} \in R$. Then $\beta^2 Z_1 \subseteq Z$. Define $Z_1^0 = \{\alpha^2 \colon \alpha \in Z_1\}$. Z_1^0 is an additive subgroup of C. For $\alpha \in Z_1$, $\beta^4 \alpha^2 = (\beta^2 \alpha)^2 \in Z^0$. Hence $\beta^4 Z_1^0 \subseteq Z^0$.

Let A consist of those elements

$$a = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in R$$

such that $\alpha_{11} = \alpha_{22}$ and $\alpha_{12}, \alpha_{21} \in Z_1^0$. A is obviously an additive subgroup of R. Since $\beta \in Z \subseteq Z_1$ and $\beta^2 e_{12} \in R$, $\beta^2 e_{12} \in A$. So A is noncentral. We show that A is invariant under all special automorphisms of R. Let

$$t = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \in R$$

be such that $t^2 = 0$. Then $\operatorname{tr}(t) = \tau_{11} + \tau_{22} = 0$ and $\det(t) = \tau_{11}\tau_{22} + \tau_{12}\tau_{21} = 0$. Let

$$a = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in A.$$

Then $\alpha_{11} = \alpha_{22}$ and $\alpha_{12}, \alpha_{21} \in Z_1^0$. We compute $(1 + t)a(1 + t) = a + [a, t] + tat$ as follows: Using the fact that $\alpha_{11} = \alpha_{22}$ and $\tau_{11} = \tau_{22}$,

$$[a, t] = \left[\begin{pmatrix} 0 & \alpha_{12} \\ \alpha_{21} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \tau_{12} \\ \tau_{21} & 0 \end{pmatrix}\right] = \begin{pmatrix} \alpha_{12}\tau_{21} + \alpha_{21}\tau_{12} & 0 \\ 0 & \alpha_{12}\tau_{21} + \alpha_{21}\tau_{12} \end{pmatrix} \in A.$$

Using the fact that $\alpha_{11} = \alpha_{22}$ and $t^2 = 0$,

$$\begin{aligned} tat &= \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \begin{pmatrix} 0 & \alpha_{12} \\ \alpha_{21} & 0 \end{pmatrix} \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \\ &= \begin{pmatrix} (\alpha_{12}\tau_{21} + \alpha_{21}\tau_{12})\tau_{11} & \alpha_{12}\tau_{11}^2 + \alpha_{21}\tau_{12}^2 \\ \alpha_{12}\tau_{21}^2 + \alpha_{21}\tau_{11}^2 & (\alpha_{12}\tau_{21} + \alpha_{21}\tau_{12})\tau_{11} \end{pmatrix} \in A. \end{aligned}$$

So $(1 + t)A(1 + t) \subseteq A$ as desired.

By our assumption, A contains a proper Lie ideal, say $[I, R]$, where I is a nonzero ideal of R. Since $[I, R]C = [IC, RC] = [RC, RC]$, $[I, R]$ contains an element

$$u = \begin{pmatrix} \lambda & \gamma \\ \delta & \lambda \end{pmatrix} \qquad \text{such that } \gamma \neq 0.$$

For $\zeta \in Z$, $\zeta u \in [\zeta I, R] \subseteq [I, R] \subseteq A$. So $\zeta\gamma \in Z_1^0$. Hence $\beta^4(\zeta\gamma) \in \beta^4 Z_1^0 \subseteq Z^0$. Set $\alpha = \beta^4\gamma$. Then $\alpha \in \beta^4 Z_1^0 \subseteq Z^0$ and $\alpha Z \subseteq Z^0$ as desired.

We need some lemmas before proceeding to prove that (2) $\Rightarrow$ (1)

LEMMA 8. *Suppose that U is a noncentral C-subspace of C_2 invariant under all special automorphisms of C_2. Then $U \supseteq [C_2, C_2]$.*

PROOF. This is contained in theorem 1.15 ([2]).

Although in the statement of the theorem, the invariance under all *inner* automorphisms is assumed, only the invariance under all special automorphisms of R is actually used in its proof.

LEMMA 9. *Suppose that U is an additive subgroup of R such that $CU \supseteq [C_2, C_2]$. Then ZU contains a proper Lie ideal of R.*

PROOF. Since $CU \supseteq [C_2, C_2]$ and since 1, e_{12}, $e_{21} \in [C_2, C_2]$, there is $\alpha \in Z$ such that $\alpha, \alpha e_{12}, \alpha e_{21} \in ZU$. Suppose that

$$r = \sum_{i,j=1}^{2} \gamma_{ij} e_{ij} \in [R, R].$$

Then $\gamma_{11} = \gamma_{22}$. Write $r = \gamma_{11}1 + \gamma_{12}e_{12} + \gamma_{21}e_{21}$. Choose $\beta \in Z$ such that $\beta e_{ij} \in R$ for $1 \leqq i, j \leqq 2$. As in the proof of (1) $\Rightarrow$ (2), $\beta^2\gamma_{11}, \beta^2\gamma_{12}, \beta^2\gamma_{21} \in Z$. So

$$\alpha\beta^2 r = (\beta^2\gamma_{11})(\alpha 1) + (\beta^2\gamma_{12})(\alpha e_{12}) + (\beta^2\gamma_{21})(\alpha e_{21}) \in Z(ZU) \subseteq ZU.$$

Hence ZU contains the proper Lie ideal $\alpha\beta^2[R, R]$.

As before, we assume that A is a noncentral additive subgroup invariant under all special automorphisms of R.

LEMMA 10. *CA is a noncentral C-subspace of C_2 invariant under all special automorphisms of C_2.*

PROOF. Let $a \in A$ and $t \in C_2$ be such that $t^2 = 0$. It suffices to show that $(1+t)a(1+t) \in AC$. Obviously, there exists $\alpha \in Z\backslash\{0, 1\}$ such that $\alpha t \in R$. Then

$$(1+\alpha t)a(1+\alpha t) - a = \alpha[a, t] + \alpha^2 tat \in A.$$

Since $\alpha^2 t = \alpha(\alpha t) \in R$, we also have

$$(1+\alpha^2 t)a(1+\alpha^2 t) - a = \alpha^2[a,t] + \alpha^4 tat \in A.$$

Using the fact that $\alpha \neq 0, 1$, we can solve $[a,t]$, $tat \in CA$.

Combining Lemmas 8, 9, 10, we have the following

LEMMA 11. *ZA contains a proper Lie ideal of R.*

For $\zeta \in Z$, we define $A(\zeta) = \{a \in A : \zeta a \in A\}$.

LEMMA 12. *Assume that Z is a fractionary ideal of Z^0. If $A(\zeta) \not\subseteq Z$ for some $\zeta \in Z \setminus \{0,1\}$, then A contains a proper Lie ideal of R.*

PROOF. Let $a \in A(\zeta)$ and $t \in T$. Then

$$\zeta(at + ta + tat) = (1+t)(\zeta a)(1+t) - \zeta a \in A.$$

Also,

$$\zeta(at + ta) + \zeta^2 tat = (1+\zeta t)a(1+\zeta t) - a \in A.$$

Subtracting the second from the first, $(\zeta^2 + \zeta)tat \in A$. Let B be the additive subgroup generated by $(\zeta^2+\zeta)tat$, where $a \in A(\zeta)$ and $t \in T$. B is obviously invariant under all special automorphisms of R. We show that B is noncentral. Suppose otherwise. Then for $t \in T$, $a \in A(\zeta)$, $tat \in Z \cap T = \{0\}$. Let e be a nontrivial idempotent of RC. Pick a nonzero ideal I of R such that $eI, Ie, eIe \subseteq R$. For $x \in I$, we have $ex(1-e)aex(1-e) = 0$. Hence $(1-e)ae = 0$. (See lemma 2 [8], for example.) Similarly, we also have $ea(1-e) = 0$. So $[a,e] = 0$. By Lemma 2, $a \in Z$. So $A(\zeta) \subseteq Z$. This is absurd.

For $\delta \in Z$, $\delta^2(\zeta^2+\zeta)tat = (\zeta^2+\zeta)(\delta t)a(\delta t) \in B$. So $Z^0 B \subseteq B$. Since Z is assumed to be a fractionary ideal of Z^0, there exists $0 \neq \alpha \in Z^0$ such that $\alpha Z \subseteq Z^0$. Consider αB. Since B is noncentral and invariant under all special automorphisms of R, so is αB. By Lemma 11, $Z(\alpha B)$ contains a proper Lie ideal of R. Observe that

$$Z(\alpha B) = (\alpha Z)B \subseteq Z^0 B \subseteq B.$$

Hence B contains a proper Lie ideal of R and so does A as desired.

Now we are ready to give

PROOF OF (2) $\Rightarrow$ (1). Suppose that Z is a fractionary ideal of Z^0. Assume on the contrary that A does *not* contain proper Lie ideals. We want to show $A \subseteq Z$. Since $C \neq \{0,1\}$, $Z \neq \{0,1\}$. Pick $\zeta \in Z \setminus \{0,1\}$. Then $\zeta^2 \notin \{0,1\}$. By Lemma 12, $A(\zeta^2) \subseteq Z$.

Let

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Choose a nonzero ideal I of R such that $eI(1-e) \subseteq R$, $(1-e)Ie \subseteq R$, $\zeta eI(1-e) \subseteq R$ and $\zeta(1-e)Ie \subseteq R$. Suppose that

$$t = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}$$

are nonzero elements of $eI(1-e)$ and $(1-e)Ie$ respectively. By our choice of I, s, t, ζs, ζt are all in R.

Let

$$a = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

be an arbitrary element of A. We compute

$$b_1 = at + ta + tat = \begin{pmatrix} \beta z & \beta(x+w) + \beta^2 z \\ 0 & \beta z \end{pmatrix} \in A,$$

$$b_2 = \zeta(at + ta) + \zeta^2 tat = \begin{pmatrix} \zeta\beta z & \zeta\beta(x+w) + \zeta^2\beta^2 z \\ 0 & \zeta\beta z \end{pmatrix} \in A.$$

We *claim* that if $\mathrm{tr}(a) = x + w = 0$ then a is central. Suppose that $x + w = 0$. By the formula above,

$$b_1 = \begin{pmatrix} \beta z & \beta^2 z \\ 0 & \beta z \end{pmatrix}, \quad b_2 = \begin{pmatrix} \zeta\beta z & \zeta^2\beta^2 z \\ 0 & \zeta\beta z \end{pmatrix}.$$

Set $c_1 = b_1 s + s b_1 + s b_1 s$ and $c_2 = b_2 s + s b_2 + s b_2 s$. Another direct computation gives

$$c_1 = \begin{pmatrix} \gamma\beta^2 z & 0 \\ \gamma^2\beta^2 z & \gamma\beta^2 z \end{pmatrix} \quad \text{and} \quad c_2 = \zeta^2 c_1.$$

Since b_1, b_2, c_1 and c_2 are all in A, we have $c_1 \in A(\zeta^2) \subseteq Z$. Hence $z = 0$. Interchanging s and t and using a similar argument, we can show that $y = 0$. So

$$a = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

is central.

For an arbitrary element

$$a = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

of A, by the formula above, $b_1 = at + ta + tat$ and $b_2 = \zeta(at + ta) + \zeta^2 tat$ both have trace zero. Applying the claim above, we have

$$\beta(x + w) + \beta^2 z = 0,$$

$$\zeta\beta(x + w) + \zeta^2\beta^2 z = 0.$$

Solving these two equations, we have $x + w = 0$ and $z = 0$. Interchanging s and t, we have also $y = 0$. So

$$a = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

is central. This completes the proof of Theorem 2.

Observe that if Z is a fractionary ideal of Z^0, then C, the quotient field of Z, must be a perfect field. Hence if C is an imperfect field of characteristic 2, then for *any* order R of C_2, there is a noncentral invariant additive subgroup which does *not* contain any proper Lie ideals. On the other hand, suppose that C is a finite field, which must be perfect. Then the only order of C_2 is C_2 itself. Hence Theorem 1 holds if C is finite and $C \neq \{0, 1\}$.

To conclude this paper, we remark that one can replace *proper* Lie ideals by *noncentral* Lie ideals in both Theorem 1 and Theorem 2 without affecting their validity and strength. On the one hand, proper Lie ideals are obviosly noncentral. On the other hand, it is proved in theorem 13 [6] that any noncentral Lie ideal of R is proper unless $\operatorname{ch} R = 2$ and $\dim_C RC \leqq 4$. Since it is assumed that $\operatorname{ch} R \neq 2$ or $\dim_C RC > 4$ in Theorem 1, proper Lie ideals and noncentral Lie ideals are really the same there. For Theorem 2, suppose that A contains a noncentral Lie ideal U, which might be improper. Let V be the additive subgroup generated by $[U, R]$ and all $(1 + t)[U, R](1 + t)^{-1}$ for $t \in R$ such that $t^2 = 0$. Obviously, $A \supseteq V$ and $V \supseteq ZV$. If V is central, then $[U, R] \subseteq Z$ and this implies immediately $U \subseteq Z$, a contradiction. Since V is invariant under special automorphisms of R, by Lemma 11, ZV contains a proper Lie ideal and so does A.

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that $RC = C_2$, ch $R = 2$ and $C \neq \{0, 1\}$ was suggested by the referee. Theorem 2 was added in the revision.

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